Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00035-1
J. Appl. Maths Mechs, Vol. 65, No. 2, pp. 311–316, 2001
© 2001 Elsevier Science Ltd
All rights reserved. Printed in Great Britain
0021-8928/01/$—see front matter

A PROBLEM OF OPTIMIZING THE STRESSED STATE IN AN ELASTIC SOLID†

A. G. SOKOLOV and V. V. STRUZHANOV

Ekaterinburg

(*Received* 10 *February* 2000)

An operator equation is obtained, the solution of which is the natural (residual) stress tensor that reduces to zero the level of stresses from an external load in a specified region of an elastic solid. It is shown that the operator of this equation possesses the property of contraction. The solution is found by the method of successive approximations. Analytical and numerical examples are given. © 2001 Elsevier Science Ltd. All rights reserved.

One of the means of influencing the operating efficiency of structural components is to create fields of natural (residual) stresses in these components during their manufacture [1–3]. Depending on the working conditions, various requirements can be formulated regarding the characteristics of these fields. In particular, the following problem of optimizing the stressed state is possible: for a specified external load, it is required to find those residual stresses that, together with the stresses from the load, enable a stressed state to be obtained in individual parts of the body as close to zero as desired.

1. FORMULATION OF THE PROBLEM

After the completion of many technological operations in the manufacture of structural components, so-called initial strains appear in them that do not satisfy the conditions of compatibility [4]. In other words, material volumes free of bonds acquire a residual strain that makes it impossible to construct a continuous solid from then. Therefore, the response of a material striving to retain continuity manifests itself in the appearance in the solid of self-balanced forces which produce strains, the magnitude of which in such that, together with the initial strains, satisfies the conditions of compatibility, and the solid retains its continuity. Thus

$$\varepsilon' = \varepsilon^* + \varepsilon'' \tag{1.1}$$

where ε^* and ε'' are symmetrical second-rank tensors of the initial strains and of the strains produced by the self-balanced forces, and $\mathrm{Inc}\ \varepsilon' = 0$ (Inc is the operator of incompatibility [5]). Note that the forces mentioned are determined by the symmetric second-rank stress tensor tensor σ'', the components of which are the primary or residual stresses. They are obviously related to the components of the tensor ε'' by Hooke's law, namely $\sigma'' = C \cdot\cdot\, \varepsilon''$ ($\varepsilon'' = S \cdot\cdot\, \sigma''$). Here, C and S are symmetrical, in general, anisotropic, fourth-rank tensors of the modulus of elasticity and the compliance respectively, and the two dots denote the double scalar product of the tensors [5].

To determine the field of the natural stresses in an elastic solid with a specified field of initial strains, it is necessary to solve the boundary-value problem

$$\nabla \cdot \sigma'' = 0, \ \ \varepsilon' = \mathrm{def}\, u, \ \ \sigma'' = C \cdot\cdot (\varepsilon' - \varepsilon^*), \ \ \sigma'' \cdot n|_\Gamma = 0 \tag{1.2}$$

The first group of equations is the equilibrium equations, the second group is the Cauchy relations, the third group is the constitutive relations, which are obtained by solving Eq. (1.1) for the stresses, and the final equation is the boundary conditions when there are no external forces; n is the vector of the outward normal to the fairly smooth boundary Γ of the solid V.

An external load produces a stress–strain state in the body, which is found by solving the boundary-value problem

$$\nabla \cdot p' = 0, \ \ e' = \mathrm{def}\, u, \ \ p' = C \cdot\cdot e', \ \ p' \cdot n|_\Gamma = t \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 65, No. 2, pp. 317–322, 2001.

where p' and e' are the stress and strain tensors respectively, and t is the vector of external forces.

If in this case there are initial strains in the solid, then, according to the principle of superposition, the stressed state in it is determined by the sum of the tensors $p'(x) + \sigma''(x), x \in V$.

It will be assumed that the tensor $\sigma''(x)$ optimizes the stressed state in the elastic solid if, in some region $V_1 \subset V$ specified in advance, the following equality holds

$$\chi(x)\sigma''(x)+\chi(x)p'(x)=0, \quad \chi(x)=\{1, x\in V_1;\ 0, x\notin V_1\} \tag{1.4}$$

Hence the problem can be formulated as follows: it is required to find the field of initial strains (the tensor $\varepsilon^*(x)$) for which residual stresses (the tensor $\sigma''(x)$) arise such that, in the region V_1, Eq. (1.4) is satisfied.

2. THE PROPERTIES OF THE BASIC EQUATIONS

To solve the problem, we will first clarify some properties of systems of equations (1.2) and (1.3). Consider the Hilbert energy space T, which consists of all possible stress tensors defined in V [6]. The scalar product and the norm in it are specified by the formulae

$$(p_1, p_2)=\int_V p_1\cdot\cdot S\cdot\cdot p_2 dV, \quad \|p\|^2=(p,p), \quad p, p_1, p_2\in T$$

The space T is the orthogonal sum of the subspaces [6]

$$T_1=\{p' : p'=C\cdot\cdot e',\ \mathrm{Jnk}\, e'=0\}, \quad T_2=\{\sigma'' : \nabla\cdot\sigma''=0,\ \sigma''\cdot n\,|_\Gamma=0\}$$

Note that the solutions of boundary-value problem (1.3) for different values of t are elements of the subspace T_1, while the solutions of problem (1.2) for different ε^* are elements of the subspace T_2.

We will represent the boundary-value problem (1.2) in the form of the operator equation $\sigma'' = A\sigma^*$, where $\sigma^* = C\cdot\cdot\varepsilon^*$ is the pseudostress tensor. The linear operator A is the difference of operators $A = B - \Lambda$, where Λ is the identity operator and B is defined by the system of equations

$$\nabla\cdot\sigma'=\nabla\cdot\sigma^*, \quad \varepsilon'=\mathrm{def}\, u, \quad \sigma'=C\cdot\cdot\varepsilon', \quad \sigma'\cdot n\,|_\Gamma=\sigma^*\cdot n\,|_\Gamma \tag{2.1}$$

i.e. $B\sigma^* = \sigma'$.

Theorem 1. The solution of system (2.1) is unique and is the orthogonal projection of the element $\sigma^* \in T$ onto the subspace T_1.

Proof. It is obvious that the solution of system (2.1) is a certain tensor $\sigma' \in T_1$. We will find the coefficients of the expansion of this tensor in a Fourier series in the orthogonal basis $\{q_k\}$ of the subspace T_1. Using the Gauss' formula, we obtain

$$(\sigma', q_k)=\int_V \sigma'\cdot\cdot S\cdot\cdot q_k dV=\int_V \sigma'\cdot\cdot\mathrm{def}\, v_k\, dV=-\int_V \nabla\cdot\sigma'\cdot v_k\, dV+\int_\Gamma n\cdot\sigma'\cdot v_k\, d\Gamma=$$

$$=-\int_V \nabla\cdot\sigma^*\cdot v_k\, dV+\int_\Gamma n\cdot\sigma^*\cdot v_k\, d\Gamma=\int_V \sigma^*\cdot\ \mathrm{def}\, v_k\, dV=(\sigma^*, q_k)$$

Hence

$$\sigma'=\sum_{k=1}^{\infty}(\sigma', q_k)q_k=\sum_{k=1}^{\infty}(\sigma^*, q_k)q_k=P_1\sigma^*$$

where P_1 is the operator of orthogonal projection onto the subspace T_1. Thus, $B = P_1$.

Further, suppose there are two solutions $\sigma_1' \in T_1$ and $\sigma_2' \in T_1$. Then $\sigma = \sigma_1' - \sigma_2' \in T_1$, whereas $\sigma \in T_2$, since $\nabla\cdot\sigma = 0$ and $\sigma\cdot n\,|_\Gamma = 0$. Consequently, $\sigma \in T_1 \cap T_2$. However, $T_1 \cap T_2 = 0$ and hence $\sigma = 0$ and $\sigma_1' = \sigma_2'$.

Theorem 2. The solution of system (1.2) is the orthogonal projection of the element $\sigma^* \in T$ onto the subspace T_2, taken with a minus sign.

Proof. We have $A = B - \Lambda = P_1 - (P_1 + P_1) = -P_2$. Then $\sigma = -P_2\sigma^*$. Here P_2 is the orthoprojector onto the subspace T_2.

Corollary 1. Since $P_2 = \Lambda - P_1$, we have

$$\sigma'' = -P_2\sigma^* = P_1\sigma^* - \sigma^* = \sum_{k=1}^{\infty} (\sigma^*, q_k)q_k - \sigma^*$$

Corollary 2. For the solid to be free of natural stresses, it is necessary and sufficient that $\sigma^* \in T_1$, i.e. $\operatorname{Inc} \varepsilon^* = 0$.

3. METHOD OF SOLUTION

Equation (1.4) can now be written in the form $\chi P_2\sigma^* = \chi p'$ or, taking into account the fact that $P_2 = \Lambda - P_1$,

$$\chi\sigma^* = \chi P_1\sigma^* - \chi p'$$

We will seek the solution of this equation in the set χT_1. If $\sigma^* \in \chi T_1$, then $\chi\sigma^* = \sigma^*(\chi\chi = \chi)$. Then, finally

$$\sigma^* = \chi P_1\sigma^* + \chi p' \tag{3.1}$$

Let us estimate the norm of the operator χP_1. We have $\|\chi P_1\| \leqslant \|\chi\| \, \|P_1\|$. It is well known [7] that $\|P_1\| = 1$. Further

$$\|\chi p\|^2 = \int_V \chi p \cdot\cdot S \cdot\cdot \chi p \, dV = \int_{V_1} p \cdot\cdot S \cdot\cdot p \, dV < \int_V p \cdot\cdot S \cdot\cdot p \, dV = \|p\|^2$$

where $p \cdot\cdot S \cdot\cdot p$ is a positive-definite quadratic form, and p is a certain element from the set $M \subset T$ which comprises tensors defined in the region V'', where V'' is any region in or coinciding with V, where $V_1 \subset V''$. Then, regarding χ as an operator acting from $M \subset T$ into χT, we obtain $\|\chi\| < 1$. If, however, χ acts from χT into χT, then $\|\chi\| = 1$. The operator P_1 reflects any elements from T into M apart from the elements $\chi p' \in T_1$. It follows that the operator χP_1, defined in the set $\chi T_1 \backslash \chi T_1 \cap T_1$, has $\|\chi P_1\| < 1$. Therefore, it is a contraction operator and the solution of Eq. (3.1) can be represented by a converging series

$$\sigma^* = \sum_{n=0}^{\infty} (\chi P_1)^n \chi p', \quad \sigma^* \in \chi T_1 \tag{3.2}$$

The required field of initial strains is then defined by the tensor $\varepsilon^* = S \cdot\cdot \sigma^*$.

Remarks 1. The condition $\chi p' \in T_1$ is satisfied when the region V_1 intersects the solid V, i.e. the region $V \backslash V_1$ is not a simply connected region.

2. Since $P_2(\sigma^* + \sigma') = P_2\sigma^*$, where σ is an arbitrary element of T_1, any tensor $\varepsilon = \varepsilon^* + \varepsilon'$, where $\varepsilon' = S \cdot\cdot \sigma'$, also initiates a field of residual stresses possessing property (1.4). Consequently, it is possible to adjust the field of initial strains in such a way that its parameters are technologically simpler to realize.

3. If it is necessary to create a specified field of residual stresses $\sigma''(x)$ in the solid, the field of initial strains is determined by the tensor $\varepsilon^* = \varepsilon'' + \varepsilon'$, where $\varepsilon'' = S \cdot\cdot \sigma''$, and ε' is an arbitrary tensor of the combined strains.

4. EXAMPLES

Example 1. Suppose a thin circular disc is stretched by a load q, uniformly distributed around its rim. It is required to find the initial strains which give rise to residual stresses together with the stresses from the external load, give a zero stressed state in the central zone of radius R_1.

The solutions of boundary-value problems (1.3) and (1.2) are given respectively by the formulae

$$p_r' = p_\theta' = q, \quad e_r' = e_\theta' = q\frac{1-\nu}{E} \tag{4.1}$$

$$\sigma''_r=\sigma''_\theta=-\frac{1}{2}\varepsilon^* E\frac{R^2-R_1^2}{R^2},\qquad \varepsilon'_r=\varepsilon'_\theta=\frac{1}{2}\varepsilon^*\left[1+\nu+(1-\nu)\frac{R_1^2}{R^2}\right],\ 0\le r\le R_1$$
$$\sigma''_{\theta,r}=\frac{1}{2}\varepsilon^* E R_1^2\left(\frac{1}{R^2}\pm\frac{1}{r^2}\right),\qquad \varepsilon'_{\theta,r}=\frac{1}{2}\varepsilon^* R_1^2\left[\frac{1-\nu}{R^2}\pm\frac{1+\nu}{r^2}\right],\ R_1\le r\le R \tag{4.2}$$

The subscripts r and θ denote the radial and tangential stresses and strains, E is Young's modulus, ν is Poisson's ratio, and $\varepsilon^*=\varepsilon_r^*+\varepsilon_\theta^*$ are the initial strains in the central zone. Using the procedure described above, taking account of formulae (4.1) and (4.2) and the equality $P_1 = B$, we obtain

$$\chi\sigma^*=\chi\sigma_r^*=\chi\sigma_\theta^*=\chi q\sum_{n=0}^{\infty}\left[\frac{1}{2}(1+\nu)+\frac{1}{2}(1-\nu)\frac{R_1^2}{R^2}\right]^n=2\chi q\left[(1-\nu)\left(1-\frac{R_1^2}{R^2}\right)\right]^{-1}$$

It follows that $\varepsilon^* = 2q(1-R_1^2/R^2)^{-1}E^{-1}$. Then

$$\sigma''_r=\sigma''_\theta=-q,\ 0\le r\le R_1;\ \sigma''_{\theta,r}=\frac{qR_1^2}{R^2-R_1^2}\left(1\pm\frac{R^2}{r^2}\right),\ R_1\le r\le R$$

and the overall stresses in the central zone are equal to zero, while in the ring surrounding this region we obtain the well-known solution of the Lamé problem [8].

Example 2. Consider a thin rectangular plate 160 mm wide and 200 mm high, stretched along the y axis by a uniformly distributed load of unit intensity. Figure 1 shows one-quarter of the plate. In five regions measuring 5 mm × 5mm, denoted by the numeral 6 and situated in the centre and symmetrical about the axes, it is required to find the initial strains which cause the appearance of residual stresses that, in the regions indicated, together with the stresses from stretching, give a zero value. The centre of region 6, situated in Fig. 1 closer to the edges of the plate, has the coordinates $x = 32$ mm, $y = 107$ mm.

The solutions of boundary-value problems (1.2) and (1.3) in the present example were found by the finite-element method. Carrying out calculations using the procedure described above, we obtain in the central region $\varepsilon_x^* = -1.241\times10^{-5}$, $\varepsilon_y^* = 7.718\times10^{-5}$ and $\varepsilon_{xy}^* = -1.048\times10^{-6}$, and in the remaining quadrangles $\varepsilon_x^* = -1.365\times10^{-5}$, $\varepsilon_y^* = 8.121\times10^{-5}$ and $\varepsilon_{xy}^* = -41\times10^{-6}$. The isolines of the intensity of natural stresses σ_i'' [8] are shown in Fig. 1, where the numbers by the curves correspond to the isolines with the corresponding value of σ_i'' in kg/mm^2.

Figure 2 shows isolines of the intensity of the overall stresses, where the numbers by the curves correspond to the σ_i values in kg/mm^2.

Example 3. Consider the same plate under the same load, only with a central circular hole of radius 3.5 mm. A fragment of the plate in the region of the concentrator is shown in Figs 3 and 4. In two zones positioned symmetrically at the points of stress concentrations (Fig. 4, region A of width 9 mm and height 0.5 mm), it is required to calculate the initial strains which initiate internal stresses that, together with the stresses from the external load, give zero

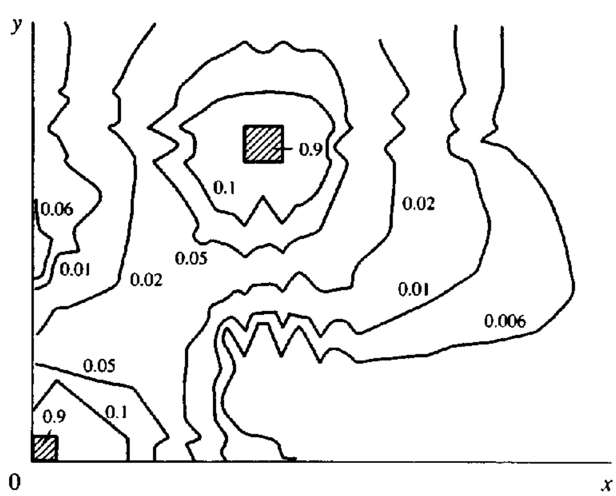

Fig. 1

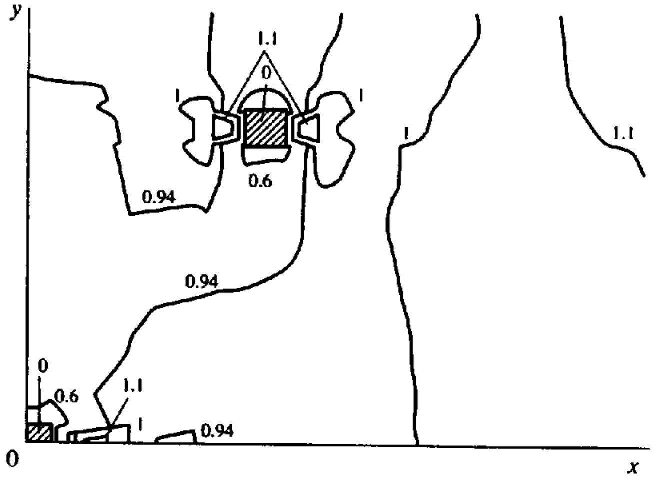

Fig. 2

values in these zones. Again, we carry out calculations using the finite-element method, region A being divided into three equal quadrangular elements. As a result, we obtain in the element adjoining the circular boundary $\varepsilon_x^* = -3.7194 \times 10^{-5}$, $\varepsilon_y^* = 1.3215 \times 10^{-3}$, and $\varepsilon_{xy}^* = 1.3598 \times 10^{-5}$, in the middle element $\varepsilon_x^* = -1.9404 \times 10^{-5}$, $\varepsilon_y^* = 1.0623 \times 10^{-3}$ and $\varepsilon_{xy}^* = 2.4654 \times 10^{-5}$ and in the final element $\varepsilon_x^* = 4.2409 \times 10^{-6}$, $\varepsilon_y^* = 5.6635 \times 10^{-6}$, and $\varepsilon_{xy}^* = 3.2819 \times 10^{-6}$.

Figure 3 shows isolines of the intensity of natural stresses, and Fig. 4 shows isolines of the intensity of the overall stresses. The numbers by the curves correspond to values of the stress intensity in kg/mm^2.

Figure 5 shows the change in the intensity of the overall stresses along the x axis. Curve *1* is the initial solution, and curves *2* and *3* are the overall stresses in cases when, to determine the primary strains, we used 2 and 40 terms of series (3.2). From the form of curve *2* we can conclude that it is possible to calculate the field of residual stresses that facilitates equalization of the stresses and considerably reduces their concentration.

Remark 4. Using the exact solution, it is possible to determine the components of the initial strain tensor that have the main influence on the characteristics of the field of residual stresses. For example, in problems 2 and 3 this is the component ε_y^*. The realization of positive residual strains ε_y^* in the optimization zones will obviously lead to the appearance in them of compressive residual stresses opposite in sign to the stresses from the external load. As a result, the stress level there will be reduced considerably, although not to zero.

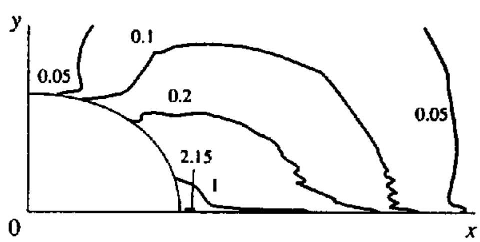

Fig. 3

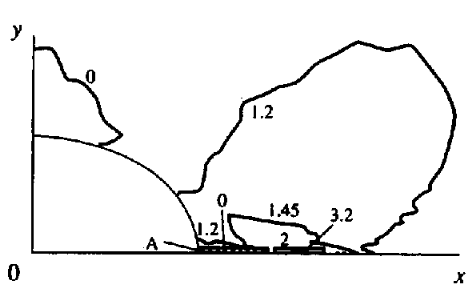

Fig. 4

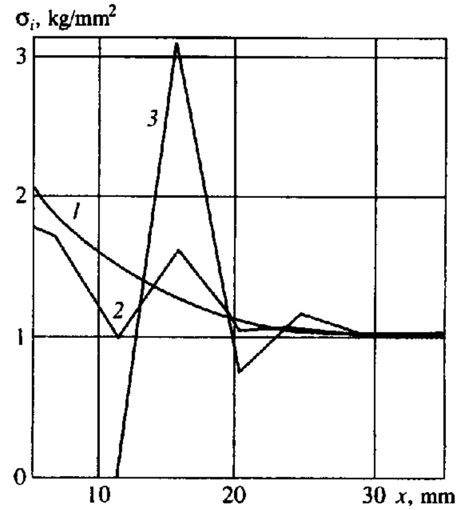

Fig. 5

In practice, methods for producing such strains are well known. One of them is the surface plastic deformation method, where, at a given point, plastic cold working is carried out. For example, for plates with a circular hole, roller burnishing of the surface of the hole with a cylindrical tool is employed to induce plastic strain, as a result of which surface compressive stresses appear.

Remark 5. The isolines given in Figs 1–4 serve to illustrate the qualitative nature of the stress distribution outside the optimization zones. They were plotted using a fairly crude piecewise-linear approximation. Smoothing algorithms were not used. Therefore, breaks appear in the isolines at points where fairly sudden change in stresses occur.

REFERENCES

1. POZDEYEV, A. A., NYASHIN, Yu. I. and TRUSOV, P. V., *Residual Stresses: Theory and Applications*. Nauka, Moscow, 1982.
2. BIRGER, I. A., *Residual Stresses*. Mashgiz, Moscow, 1963.
3. KUDRYAVTSEV, I. V., *Internal Stresses as a Strength Reserve in Engineering*. Mashgiz, Moscow, 1951.
4. TIMOSHENKO, S. P. and GOODIER, J. N., *Theory of Elasticity*. McGraw-Hill, New York, 1970.
5. LUR'YE, A. I., *Theory of Elasticity*. Nauka, Moscow, 1970.
6. MIKHLIN, S. G., *Variational Methods in Mathematical Physics*. Nauka, Moscow, 1970.
7. KANTOROVICH, L. V. and AKILOV, G. P., *Functional Analysis*. Nauka, Moscow, 1984.
8. BEZUKHOV, N. I., *Principles of the Theory of Elasticity*, Plasticity and Creep. Vysshaya Shkola, Moscow, 1968.

Translated by P.S.C.